10 Back to planar nonlinear systems

10.1 Near the equilibria

Recall that I started talking about the Lotka–Volterra model as a motivation to study systems of two first order autonomous equations of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^2,\ \boldsymbol{f}\colon U \longrightarrow \mathbf{R}^2. \tag{1}$$

After this I discussed general properties of systems of the form (1) and formulated the main goal: Given the system how can I obtain its phase portrait? This task can be fully solved for linear systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(t) \in \mathbf{R}^2,\ \boldsymbol{A} = (a_{ij})_{2\times 2}. \tag{2}$$

In this lecture I will show how the knowledge of the phase portraits of (2) can be used to obtain partial and *essentially local* information about phase portraits of (1).

The general idea is quite straightforward. Assume that system (1) has an equilibrium $\hat{\boldsymbol{x}}$ (i.e., $\boldsymbol{f}(\hat{\boldsymbol{x}}) = 0$) and expand $\boldsymbol{f}$ in its Taylor series around this point:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}}) + \boldsymbol{f}'(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}}) + \mathcal{O}\big(|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2\big) = \boldsymbol{f}'(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}}) + \mathcal{O}\big(|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2\big),$$

where

$$\boldsymbol{f}'(\hat{\boldsymbol{x}}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{(x_1,x_2)=(\hat{x}_1,\hat{x}_2)}$$

is called the *Jacobi* (or *Jacobian*) matrix of $\boldsymbol{f}$ evaluated at the point $\hat{\boldsymbol{x}}$. There is a temptation to drop the terms $\mathcal{O}\big(|\boldsymbol{x}-\hat{\boldsymbol{x}}|^2\big)$ and make the shift of the variables $\boldsymbol{y} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Then for $\boldsymbol{y}$ I obtain the linear system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}'(\hat{\boldsymbol{x}})\boldsymbol{y} = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{A} := \boldsymbol{f}'(\hat{\boldsymbol{x}}), \tag{3}$$

whose phase portrait I know. System (3) is called the *linearization* of (1) around (or at) the equilibrium $\hat{\boldsymbol{x}}$. The answer when I can use the linearization to figure out the behavior near $\hat{\boldsymbol{x}}$ is given by the following theorem.

Theorem 1 (Grobman–Hartman)**.** *Assume that the origin of the linearization of* (1) *at* $\hat{\boldsymbol{x}}$ *is hyperbolic. Then system* (1) *is locally topologically equivalent to its linearization* (3)*.*

Note that the theorem holds only in some neighborhood of $\hat{\boldsymbol{x}}$. To extend the terminology from the linear system, I call the equilibrium $\hat{\boldsymbol{x}}$ *hyperbolic*, if the Jacobi matrix $\boldsymbol{f}'$ evaluated at this equilibrium has no eigenvalues with zero real part. In particular, denoting d_-, d_+, d_0 the number of eigenvalues with negative, positive, and zero real parts, I have

Theorem 2 (Lyapunov, Poincaré)**.** *If* $\hat{\boldsymbol{x}}$ *hyperbolic, then it is asymptotically stable if* $d_- = d$*. If* $d_+ > 0$ *for any* $\hat{\boldsymbol{x}}$ *(including non-hyperbolic equilibria) then* $\hat{\boldsymbol{x}}$ *is unstable.*

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If the equilibrium is non-hyperbolic in the linear system, for the analysis of the orbit structure of the nonlinear equation I will need to use some additional tools. This is in particular true when the linearization is a center. For example, for the Lotka–Volterra model the nontrivial equilibrium is a center in the linearized system (check this), and how I argued, it is also a center in the nonlinear system. However, according to the general theory, I could not conclude that the equilibrium is a center because the linearization is a center!

Therefore, for hyperbolic equilibria of (1) exactly the same classification of the phase portraits in *some neighborhood* of $\hat{\boldsymbol{x}}$ holds, as for linear systems. If a point is hyperbolic, then we will see familiar nodes, spirals, and saddles. If I deal with saddles, even more can be said.

Let me define first the stable and unstable manifolds of $\hat{\boldsymbol{x}}$. By definition, the *stable manifold* of $\hat{\boldsymbol{x}}$ is defined as

$$W_-(\hat{\boldsymbol{x}}) := \{\boldsymbol{x}_0 \in U \colon \boldsymbol{x}(t;\boldsymbol{x}_0) \to \hat{\boldsymbol{x}} \text{ as } t \to \infty\},$$

and the *unstable manifold* of $\hat{\boldsymbol{x}}$ is defined as

$$W_+(\hat{\boldsymbol{x}}) := \{\boldsymbol{x}_0 \in U \colon \boldsymbol{x}(t;\boldsymbol{x}_0) \to \hat{\boldsymbol{x}} \text{ as } t \to -\infty\}.$$

Theorem 3. *Assume that $\hat{\boldsymbol{x}}$ is a hyperbolic and denote T_-, T_+ the stable and unstable subspaces of its linearization. If $T_- \neq \emptyset$ then $W_-(\hat{\boldsymbol{x}})$ exists and is tangent to T_- at $\hat{\boldsymbol{x}}$, and if $T_+ \neq \emptyset$ then $W_+(\hat{\boldsymbol{x}})$ exists and is tangent to T_+ at $\hat{\boldsymbol{x}}$.*

The last theorem means that the saddle structure of the equilibrium of the linearization is preserved in the nonlinear system, i.e., there are orbits that approach the equilibrium for $t \to \infty$, and there are orbits that approach the equilibrium for $t \to -\infty$.

To illustrate the last theorem consider a simple example

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= y + x^2. \end{aligned}$$

This system has a unique equilibrium $\hat{\boldsymbol{x}} = (0, 0)$, and the linearization at this point is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}.$$

(I use here and in the following, with a slight abuse of notations, the same variable to denote the linearization of the system). The linearization is a saddle point with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and eigenvectors $\boldsymbol{v}_1 = (1, 0)^\top$ and $\boldsymbol{v}_2 = (0, 1)^\top$ respectively, therefore the stable subspace T_- is the x_1-axis and unstable subspace T_+ is the x_2-axis. For the full nonlinear system I also have the same orbit structure on x_2-axis (because if $x_1 = 0$ then the system is linear), and the simple form of the equations allows me to find that the orbits on the plane (x, y) are given by

$$y = -\frac{x^3}{3} + \frac{C}{x}\,.$$

Therefore for $C = 0$ I have orbits approaching the origin along the parabola

$$y = -\frac{x^3}{3}\,,$$

which is the stable manifold for my equilibrium, and which is obviously tangent to T_- (see the figure).

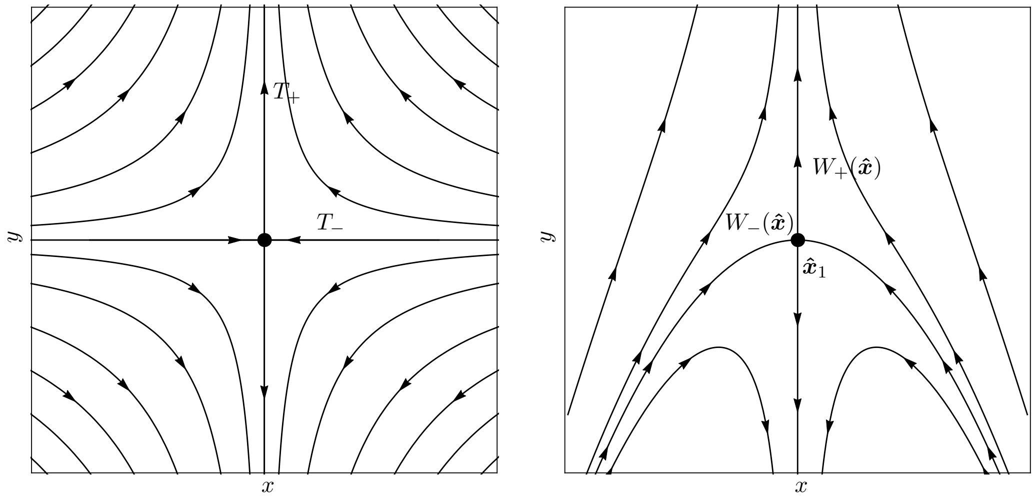

Figure 1: Stable and unstable manifolds for a hyperbolic equilibrium

10.2 Outside of equilibria

The previous subsection tells us a lot of possible orbit behavior near equilibria. The natural question is of course how to get an idea about what happens outside of equilibria. Right at this point I would like to say that there are no universal methods to analyze the structure of the phase portrait of a non-linear ODE system, however, by now we have quite a rich arsenal of tools to extract at least partial knowledge on the orbit behavior.

For the planar system

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \end{aligned} \tag{4}$$

the essential fact is that a simple curve dissects $\mathbf{R}^2$ into two connected regions. I can use this fact to identify the regions of $\mathbf{R}^2$, where the solutions to (4) are monotone (i.e., the derivatives have definite signs in these regions). Such regions are found by plotting the *null-clines*

$$f(x,y) = 0, \quad g(x,y) = 0$$

and using the following almost obvious proposition

Proposition 4. *Let $\boldsymbol{\phi}(t) = \big(x(t), y(t)\big)$ be a solution to (4). Consider an open bounded set $V \subseteq \mathbf{R}^2$. If $x(t)$ and $y(t)$ are strictly monotone in V then either $\boldsymbol{\phi}(t)$ hits the boundary of V at some finite t, or $\boldsymbol{\phi}(t)$ converges to an equilibrium $(\hat{x}, \hat{y}) \in \overline{V}$.*

Example 5. Consider the system

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= 1 - x^2 - y^2. \end{aligned}$$

The Jacobi matrix is

$$\boldsymbol{f}'(x,y) = \begin{bmatrix} -1 & 0 \\ -2x & -2y \end{bmatrix}.$$

The system has two equilibria: $\hat{\boldsymbol{x}}_1 = (0, 1)$ and $\hat{\boldsymbol{x}}_2 = (0, -1)$. The first one is an asymptotically stable node since

$$\boldsymbol{f}'(\hat{\boldsymbol{x}}_1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

and the eigenvalues are -1 and -2. Therefore there exists a two-dimensional stable manifold $W_-(\hat{\boldsymbol{x}}_1)$. The second equilibrium is unstable because

$$\boldsymbol{f}'(\hat{\boldsymbol{x}}_2) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

with the stable manifold tangent to the line parallel to x-axis and unstable manifold tangent (actually, coincident with) to y-axis.

The null-clines of the system are given by

$$l_1 = \{(x, y) \colon x = 0\}, \quad l_2 = \{(x, y) \colon x^2 + y^2 = 1\},$$

therefore there are four regions of $\mathbf{R}^2$ where the solutions are monotone (see the figure). Using the proposition above, it should be clear that the only possible orbit structure is given in the figure. In general, it is not quite straightforward to guess a possible structure of the orbits, however, the null-clines are usually quite useful for obtaining at least partial information.

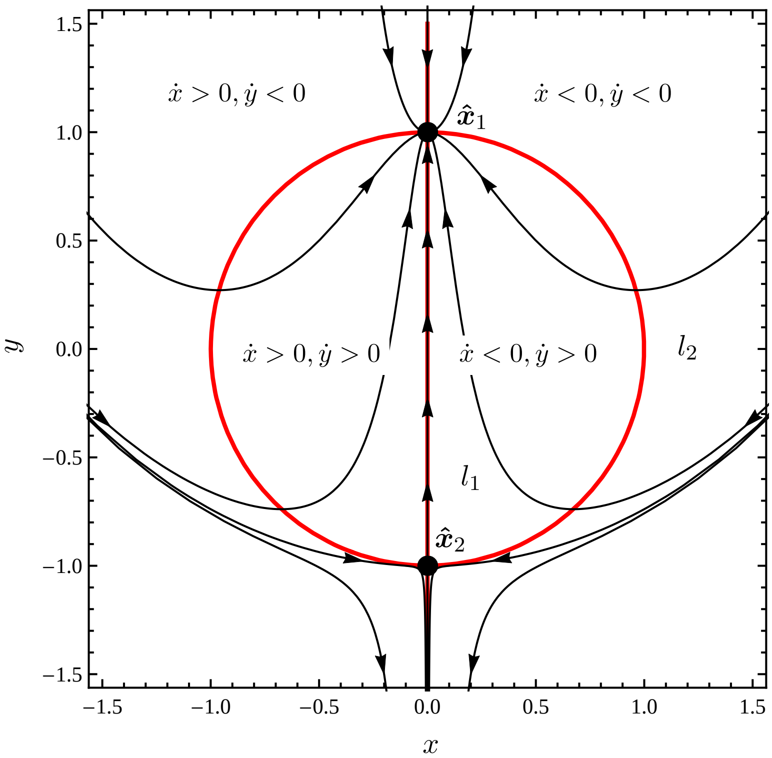

Figure 2: The phase portrait of the system in the text. The red curves are null-clines of the system

10.3 Bifurcations of equilibria. Structural stability

Recall that a *bifurcation* is the change of the topological type of the system. Equilibria can change their topological type under some parameter variation only if the linearization of the system around these equilibria is non-hyperbolic, or, in other words, a presence of a non-hyperbolic equilibria is a necessary condition for a bifurcation. This implies that bifurcations of the equilibria of the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \alpha), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^2, \quad \alpha \in \mathbf{R},$$

are possible when either one of the eigenvalues of the Jacobi matrix becomes zero, or when two eigenvalues cross the real axis. (I consider here only the case when there is one parameter in the system. In general, of course, it is possible to have that both eigenvalues of the Jacobi matrix becomes zero simultaneously, but generically for this it is necessary to have two parameters in the system.)

Example 6 (*Saddle-node* or *fold* bifurcation)**.** Consider the planar system

$$\begin{aligned} \dot{x} &= \alpha + x^2, \\ \dot{y} &= -y. \end{aligned}$$

Here the equations are decoupled and it is quite straightforward to figure out the phase portraits depending on the sign of α. If $\alpha < 0$ I have two equilibria, one is asymptotically stable (node) and the other one is unstable (saddle), if $\alpha = 0$ then the origin is a non-hyperbolic equilibrium (unstable, this equilibrium is called the saddle-node, hence the name for the bifurcation), and finally if $\alpha > 0$ then there are no equilibria in the system, see the figure.

Here in this example the bifurcation is essentially one-dimensional, which I already studied at length in the earlier lecture notes for the this course. It turns out that this situation is generic for the case when one of the eigenvalues of the Jacobi matrix becomes zero. I can have for most of the cases either fold, or transcritical, or pitchfork bifurcation, which were already studied. In the first case two equilibria of the opposite stability approach each other, collide and disappear; in the second case there is always an equilibrium for all parameter values that changes the stability properties as another equilibrium "passes" through it, and finally in the third case I should distinguish sub- and supercritical pitchfork bifurcation, this occurs in systems with certain symmetries. It is possible to specify the exact mathematical conditions for these three bifurcation types, but in practice it is usually much easier to determine the type of the bifurcation with one of the eigenvalues zero by analyzing the types and number of equilibria for the parameter values close to the bifurcation point, I will present some examples of such analysis.

On the other hand, the case when two eigenvalues of the Jacobi matrix cross the imaginary axis is essentially two-dimensional. The corresponding bifurcation is called Hopf, or, more appropriately, *Poincaré–Andronov–Hopf* bifurcation, and I will study it in more details later.

Before finishing this section, I would like to note that bifurcations of equilibria in two dimensional systems are not the only possible changes of the phase portrait that lead to topologically non-equivalent pictures. Consider the following example.

Example 7 (Heteroclinic bifurcation in a planar system)**.** Consider the system depending on parameter α

$$\begin{aligned} \dot{x} &= 1 - x^2 - \alpha x y, \\ \dot{y} &= xy + \alpha(1 - x^2). \end{aligned}$$

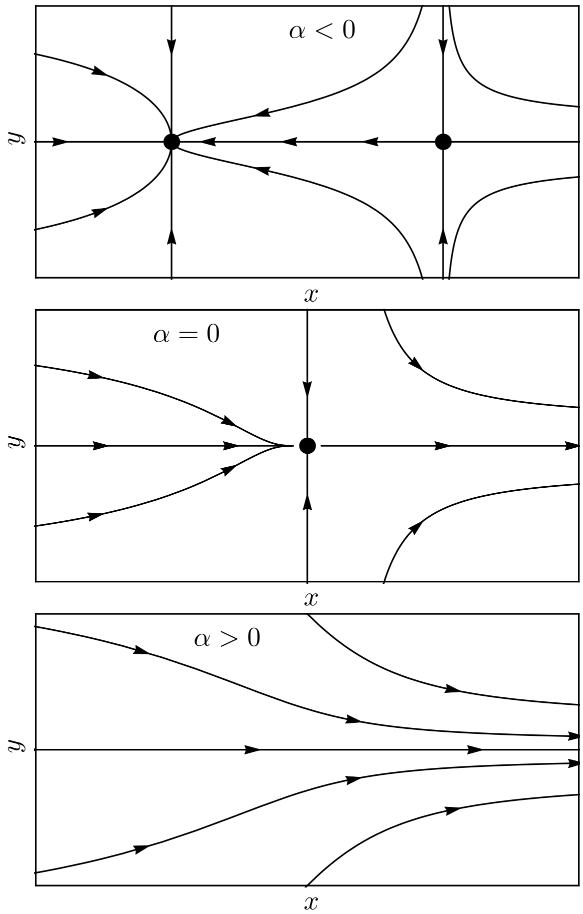

Figure 3: Saddle-node or fold bifurcation in a two dimensional system

This system always has two saddle equilibria $\hat{\boldsymbol{x}}_1 = (-1, 0)$ and $\hat{\boldsymbol{x}}_2 = (1, 0)$. At $\alpha = 0$ the x-axis is invariant and approaches both equilibria for $t \to \pm\infty$. Such trajectories are called *heteroclinic*. For $\alpha \neq 0$ this connection disappears. This is an example of a *global* bifurcation because to detect it I need to keep track of both equilibria.

There is one more important notion that pertains to bifurcations in systems of ODE. In the last example it is clear that for $\alpha = 0$ the system is such that a small system perturbation would lead to a qualitatively different phase portraits. To formalize this, consider two systems defined in a closed and bounded region U (the region is chosen as closed and bounded, i.e, compact, not to deal with complications due to unboundness of the plane):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U, \tag{5}$$

and

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U. \tag{6}$$

Now consider the $\mathcal{C}^{(1)}$ distance between these two vector fields (5) and (6) in U as

$$\|\boldsymbol{f} - \boldsymbol{g}\|_1 = \sup_{\boldsymbol{x} \in U} \big\{ |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})|, |\boldsymbol{f}'(x) - \boldsymbol{g}'(\boldsymbol{x})| \big\},$$

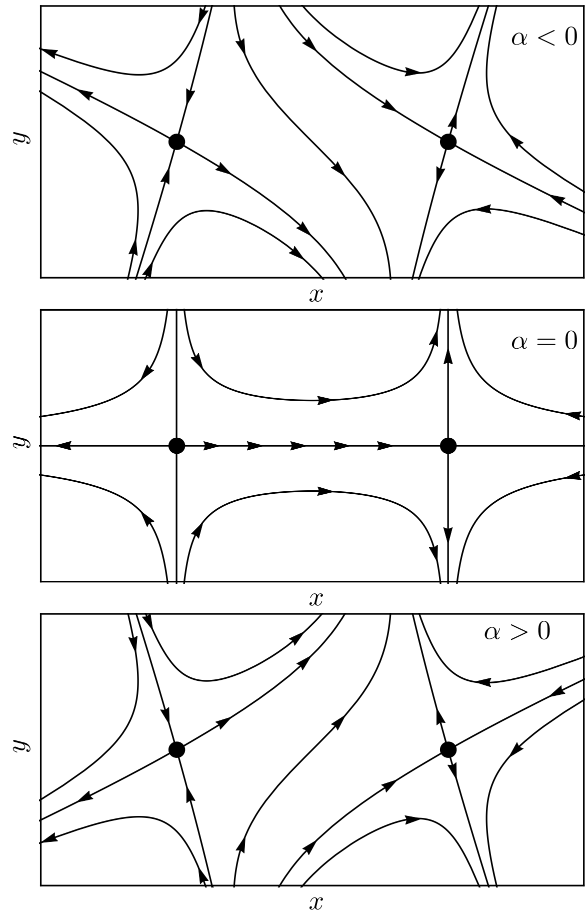

Figure 4: Global heteroclinic bifurcation

where the first norm inside the sup is the usual Euclidian norm, and the second one is the norm of a matrix $\boldsymbol{A}$. The $\mathcal{C}^{(0)}$ distance is not enough for my purposes.

Using this definition of a distance in the space of all vector fields I can define a neighborhood $\mathcal{N}(\boldsymbol{f})$ of the vector field $\boldsymbol{f}$ as a set ϵ-close to $\boldsymbol{f}$ with respect to the defined metric. Finally, I can state that

Definition 8. *A planar differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ (or, equivalently, a vector field $\boldsymbol{f}$) is called structurally stable if there is a neighborhood $\mathcal{N}(\boldsymbol{f})$ such that for any vector field $\boldsymbol{g} \in \mathcal{N}(\boldsymbol{f})$ is topologically equivalent in U to $\boldsymbol{f}$.*

One of the indication of a structurally unstable system is the presence of a non-hyperbolic equilibrium. For example, the Lotka–Volterra system is structurally unstable, which is sometimes used as an indication that this system of ODE cannot be considered as a reliable candidate for a mathematical model describing predator–prey interaction.